RESTRICTED SINGLE OR DOUBLE SIGNED PATTERNS

Toufik Mansour

Department of Mathematics,
University of Haifa, Israel 31905
tmansur@study.haifa.ac.il

Abstract

Let $E_n^r = \{[\tau]_a = (\tau_1^{(a_1)}, \dots, \tau_n^{(a_n)}) | \tau \in S_n,\ 1 \le a_i \le r\}$ be the set of all signed permutations on the symbols $1, 2, \dots, n$ with signs $1, 2, \dots, r$. We prove, for every 2-letter signed pattern $[\tau]_a$, that the number of $[\tau]_a$-avoiding signed permutations in E_n^r is given by the formula $\sum\limits_{j=0}^{n} j!(r-1)^j \binom{n}{j}^2$. Also we prove that there are only one Wilf class for $r = 1$, four Wilf classes for $r = 2$, and six Wilf classes for $r \ge 3$.

Key words: restricted permutations, pattern avoidance, signed permutations.

1. Introduction

Pattern avoidance proved to be a useful language in a variety of seemingly unrelated problems, from stack sorting [K,T] to the theory of Kazhdan-Lusztig polynomials [Br], and singularities of Schubert varieties [LS,Bi]. Signed pattern avoidance proved to be a useful language in combinatorial statistics defined in type-B noncrossing partitions, enumerative combinatorics, algebraic combinatorics, and geometric combinatorics [S,BS,M,R].

Restricted permutations. Let $\pi \in S_n$ and $\tau \in S_k$ be two permutations. An *occurrence* of τ in π is a subsequence $1 \le i_1 < i_2 < \dots < i_k \le n$ such that $(\pi_{i_1}, \dots, \pi_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that π *avoids* τ, or is τ-*avoiding*, if there is no occurrence of τ in π. The set of all τ-avoiding permutations in S_n is denoted $S_n(\tau)$. For an arbitrary finite collection of patterns T, we say that π avoids T if π avoids any $\tau \in T$; the corresponding subset of S_n is denoted $S_n(T)$.

Restricted signed permutations. We say that $(\tau_1^{(a_1)}, \dots, \tau_n^{(a_n)})$ is a *signed permutation* and denote it by $[\tau]_a$ if $(\tau_1, \dots, \tau_n) \in S_n$ and $a \in [r]^n$. In this context, we call $a_1, \dots, a_n$ the *signs* of τ, and we call $\tau_1, \dots, \tau_n$ the *symbols* of τ.

The set of all signed permutations with symbols $a_1, \dots, a_n$ and signs $d_1, d_2, \dots, d_r$ we denote by $E_{a_1,\dots,a_n}^{d_1,\dots,d_r}$; also we denote $E_n^r = \{[\tau]_a | \tau \in S_n, 1 \le a_i \le r\}$. Clearly,

by definitions $|E_n^r| = n! \cdot r^n$.

Similarly to the symmetric group S_n which is generated by the adjacent transpositions σ_i for $1 \leq i \leq n$, where σ_i interchanges positions i and $i+1$ (see also the hyperoctahedral group B_n [S]), the set E_n^r is a group which is generated by the adjacent transpositions σ_i for $1 \leq i \leq n$, along with σ_0 which acts on the right by increasing the first sign; that is,

$$(\tau_1^{(a_1)}, \tau_2^{(a_2)}, \ldots, \tau_n^{(a_n)})\sigma_0 = (\tau_1^{1+(a_1+1(mod\ r))}, \tau_2^{(a_2)}, \ldots, \tau_n^{(a_n)}).$$

Example 1. The set of all signed permutations with two symbols $1, 2$ and two signs $1, 2$ is the following set:

$$\begin{aligned} E_2^2 = \{ \; & (1^{(1)}, 2^{(1)}),\ (1^{(1)}, 2^{(2)}),\ (1^{(2)}, 2^{(1)}),\ (1^{(2)}, 2^{(2)}), \\ & (2^{(1)}, 1^{(1)}),\ (2^{(1)}, 1^{(2)}),\ (2^{(2)}, 1^{(1)}),\ (2^{(2)}, 1^{(2)})\}. \end{aligned}$$

Let $[\tau]_a \in E_k^r$, and $[\alpha]_b \in E_n^r$; we say that $[\alpha]_b$ *avoids* $[\tau]_a$ (or is $[\tau]_a$-avoiding) if there is no sequence of k indices, $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that the following two conditions hold:

(i) $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to τ;
(ii) $b_{i_j} = a_j$ for all $j = 1, 2, \ldots, k$.

Otherwise, we say that $[\alpha]_b$ *contains* $[\tau]_a$ (or is $[\tau]_a$-containing). The set of all $[\tau]_a$-avoiding signed permutations in E_n^r denoted by $E_n^r([\tau]_a)$, and in this context $[\tau]_a$ is called a *signed pattern*. For an arbitrary finite collection of signed patterns T, we say that $[\alpha]_b$ avoids T if $[\alpha]_b$ avoids any $[\tau]_a \in T$; the corresponding subset of E_n^r is denoted $E_n^r(T)$.

Example 2. As an example, $\Phi = (3, 2, 1)_{(1,2,2)} = (3^{(1)}, 2^{(2)}, 1^{(2)}) \in E_3^2$ avoids $(2^1, 1^1)$; that is, $\Phi \in E_3^2((2^{(1)}, 1^{(1)}))$.

Let T_1, T_2 be two subsets of signed patterns; we say that T_1 and T_2 are in the same *d-Wilf class* if $|E_n^r(T_1)| = |E_n^r(T_2)|$ for $n \geq 0$, $r \geq d$.

In the symmetric group S_n, for every 2-letter pattern τ the number of τ-avoiding permutations is one, and for every pattern $\tau \in S_3$ the number of τ-avoiding permutations is given by the Catalan number [K]. Also Simion [S] proved there are similar results for the hyperoctahedral group B_n. Here we are looking for similar results for E_n^r. We show that for every 2-letter signed pattern $[\tau]_a$ the number of $[\tau]_a$-avoiding signed permutations in E_n^r is given by $\sum\limits_{j=0}^{n} j!(r-1)^j \binom{n}{j}^2$, which generalize the results of [S] (see section 3).

The paper is organized as follows. The elementary definitions, and the symmetric operations, is treated in **section** 2, in **section** 3 we give the two relations between avoidance of patterns in S_k and avoidance of signed patterns in E_k^r, in

section 4 we represent two sets of signed patterns, and represent a bijection which gives a combinatorial geometric explanation for one of these results. In **sections** 5**,** 6 we prove the first and the second part of Main Theorem, respectively. Finally, in the **last section** we prove a combinatorial identity as a corollary of Main Theorem.

Main Theorem:
(i) For every 2-letter signed pattern $[\tau]_a$, the number of $[\tau]_a$-avoiding signed permutations in E_n^r is given by the expression: $\sum_{j=0}^{n} j!(r-1)^j \binom{n}{j}^2$.
(ii) A double restriction by 2-letter signed patterns gives one 1-Wilf class, four 2-Wilf clases, six r-Wilf clases for $r \geq 3$.

2. Symmetries on signed permutations

As on the symmetric group S_n there are two natural symmetric operations, the reversal and the complement (see [SS]), also on E_n^r we define:

(i) the *reversal* $er : E_n^r \to E_n^r$ defined by

$$er : (\alpha_1^{(u_1)}, \dots, \alpha_n^{(u_n)}) \mapsto (\alpha_n^{(u_n)}, \dots, \alpha_1^{(u_1)});$$

(ii) the *complement* $ec : E_n^r \to E_n^r$ defined by

$$ec : (\alpha_1^{(u_1)}, \dots, \alpha_n^{(u_n)}) \mapsto ((n+1-\alpha_1)^{(u_1)}, \dots, (n+1-\alpha_n)^{(u_n)});$$

(iii) and besides that, there is the *sign-complement* $es : E_n^r \to E_n^r$ defined by

$$es : (\alpha_1^{(u_1)}, \dots, \alpha_n^{(u_n)}) \mapsto (\alpha_1^{(r+1-u_1)}, \dots, \alpha_n^{(r+1-u_n)}).$$

Example 3. Let $\Phi = (1^{(1)}, 3^{(2)}, 2^{(1)}) \in E_3^2$, then $er(\Phi) = (2^{(1)}, 3^{(2)}, 1^{(1)})$, $ec(\Phi) = (2^{(1)}, 1^{(2)}, 3^{(1)})$, and $es(\Phi) = (1^{(2)}, 2^{(1)}, 3^{(2)})$.

Proposition 1. *The group* $< er, ec, es >$ *is isomorphic to* D_8.

More generally, we extend these symmetric operations to subsets of E_n^r: $g(T) = \{g(\Phi) | \Phi \in T\}$, where $g = er$, ec, or es.

Theorem 1. *Let* $T \subset E_k^r$. *For all* $n \geq 0$,

$$|E_n^r(T)| = |E_n^r(er(T))| = |E_n^r(ec(T))| = |E_n^r(es(T))|.$$

Now we define the fourth symmetric operation on E_n^r. Let us define

$$h_{\delta,n} : E_n^r \to E_n^r,$$

where $\delta \in S_r$ by $h_{\delta,n}([\alpha]_a) = [\alpha]_b$ such that $b_i = \delta_{a_i}$ for all $i = 1, 2, \dots, n$. More generally, $h_{\delta,n}(T) = \{h_{\delta,n}([\alpha]_a) | [\alpha]_a \in T\}$ for $T \subset E_n^r$.

Theorem 2. *Let $T \subset E_k^r$, $\delta \in S_r$. Then $|E_n^r(T)| = |E_n^r(h_{\delta,k}(T))|$.*

Proof. Let $[\alpha]_a \in E_n^r(T)$, so $[\alpha]_a$ is T-avoiding if and only if $h_{\delta,n}([\alpha]_a)$ is $h_{\delta,k}(T)$-avoiding. On the other hand $h_{\delta,n}$ is an invertible function. Hence the theorem holds. □

Corollary 1. *Let $T \subseteq E_k^r$, and let $\delta \in S_r$ such that $a_{b_j} = j$ for $j = 1, 2, \ldots, d$. For all $n \geq 0$, $|E_n^r(T)| = |E_n^r(h_{\delta,k}(T))|$.*

Example 4. As an example, for $r \geq 3$,

$$|E_n^r((1^{(1)}, 2^{(2)}), (1^{(2)}, 2^{(3)}))| = |E_n^r((1^{(2)}, 2^{(1)}), (1^{(1)}, 2^{(3)}))|,$$

by the symmetric operation $h_{(2,1,3,4,\ldots,r),n}$.

3. Avoidance patterns and signed patterns

We say a signed permutation $[\tau]_a \in E_k^r$ is *homogeneous* if $a_i = u$ for all $i = 1, 2, \ldots, k$ where $1 \leq u \leq r$; in this case we denote $[\tau]_a$ by $[\tau]_{(u)}$. More generally, we denote $T_{(u)} = \{[\tau]_{(u)} | \tau \in T\}$.

Theorem 3. *Let $1 \leq u \leq r$, $T \subset S_k$. For all $n \geq 0$*

$$|E_n^r(T_{(u)})| = \sum_{j=0}^{n} j!(r-1)^j |S_{n-j}(T)| \binom{n}{j}^2.$$

Proof. Immediately by definitions

$$|E_n^r(T_{(u)})| = \sum_{j=0}^{n} \binom{n}{j}^2 |E_{\{1,2,\ldots,j\}}^{\{u\}}(T_{(u)})||E_{\{j+1,\ldots,n\}}^{\{1,\ldots,u-1,u+1,\ldots,r\}}|,$$

where $E_{T_1}^{T_2}$ is the set of all signed permutations with set symbols T_1 and set signs T_2. So clearly $|E_{\{j+1,\ldots,n\}}^{\{1,\ldots,u-1,u+1,\ldots,r\}}| = (n-j)! \cdot (r-1)^{n-j}$, also $|E_{\{1,2,\ldots,j\}}^{\{u\}}(T_{(u)})| = |S_j(T)|$ by removing the sign u. Hence the theorem holds. □

Example 5. (*see* [S, Eq. 46]) For $a = 1, 2$, by Theorem 3,

$$|E_n^2((12)_{(a)}, (21)_{(a)})| = (n+1)!,$$

$$|E_n^2((12)_{(a)})| = |E_n^2((21)_{(a)})| = \sum_{j=0}^{n} j! \binom{n}{j}^2.$$

Theorem 4. *Let $r \geq 1$, $\tau \in S_k$. For all $n \geq 0$, $|E_n^r(F_\tau)| = r^n |S_n(\tau)|$, where $F_\tau = \{(\tau_1^{(v_1)}, \ldots, \tau_k^{(v_k)}) | 1 \leq v_i \leq r\}$.*

Proof. Let us define a function $f : [r]^n \times S_n(\tau) \mapsto E_n^r(F_\tau)$ by

$$f((u_1, \ldots, u_n; \alpha_1, \ldots, \alpha_n)) = (\alpha_1^{(u_1)}, \ldots, \alpha_n^{(u_n)}).$$

So $(u_1, \ldots, u_n; \alpha_1, \ldots, \alpha_n) \in [r]^n \times S_n(\tau)$ if and only if $(\alpha_1, \ldots, \alpha_n)$ avoids τ, which is equivalent to $(\alpha_1^{(u_1)}, \ldots, \alpha_n^{(u_n)})$ avoids F_τ for all $u_i = 1, 2, \ldots, r$. Hence f is a bijection, which means that the theorem holds. $\square$

Example 6. Let $T = \{(1^{(a)}, 2^{(b)}) | a, b = 1, 2, \ldots, r\}$; by Theorem 4 we obtain $|E_n^r(T)| = r^n$ for all $n \geq 0$.

4. Restricted sets

In this section, we calculate cardinalities of $E_n^r(T)$ for two special subsets $T \subset E_2^r$. The first special subset is defined by $T_{b;a_1,a_2,\ldots,a_l} = \{(1^b, 2^{(a_j)}) | j = 1, 2, \ldots, l\}$.

Theorem 5. *Let $1 \leq l \leq r$, and $1 \leq b \leq a_1 < a_2 < \ldots < a_l \leq r$. Then*

$$\sum_{n \geq 0} \frac{|E_n^r(T_{b;a_1,a_2,\ldots,a_l})|}{n!} x^n = \left(\frac{1 - (r-l)x}{(1-(r-1)x)^l} \right)^{\frac{1}{l-1}};$$

when $l = 1$ we take the limit of the right hand side which equals $\frac{e^{\frac{1}{1-(r-1)x}}}{1-(r-1)x}$.

Proof. By Corollary 1 $|E_n^r(T_{b;a_1,\ldots,a_l})| = |E_n^r(1; a, a+1, \ldots, a+l-1)|$.

Let $\Phi \in E_n^r(T_{1;a+1,\ldots,a+l-1})$, $p_r(n) = |E_n^r(T_{1;a+1,\ldots,a+l-1})|$, and let us consider the possible values of Φ_1:

1. Let $\Phi_1 = i^{(c)}$, $c \neq 1$, and $1 \leq i \leq n$; so $\Phi \in E_n^r(T_{1;a+1,\ldots,a+l-1})$ if and only if $(\Phi_2, \ldots, \Phi_n)$ is $T_{1;a+1,\ldots,a+l-1}$ -avoiding, hence in this case there are $(r-1)np_r(n-1)$ signed permutations.
2. Let $\Phi_1 = i^{(1)}$; since Φ is $T_{1;a+1,\ldots,a+l-1}$-avoiding, the symbols $i+1, \ldots, n$ appeared with sign $d \geq a+1$ or $d \leq a-1$. Also the symbols $1, \ldots, i-1$ are $T_{1;a+1,\ldots,a+l-1}$-avoiding, and can be replaced anywhere at positions $2, \ldots, n$, hence there are $\sum_{i=1}^{n} \binom{n-1}{i-1} |E_{\{i+1,\ldots,n\}}^{\{1,\ldots,a-1,a+l,\ldots,r\}}| \cdot |E_{i-1}^r(T)|$ signed permutations, which means there are $\sum_{i=1}^{n} \binom{n-1}{i-1} (n-i)!(r-l)^{n-i} p_r(i-1)$ signed permutations.

So by the above two cases we obtain a recurrence relation satisfied by p_n

$$p_n = (r-1)np_{n-1} + \sum_{i=1}^{n} \binom{n-1}{i-1} (n-i)!(r-l)^{n-i} p_{i-1},$$

for $n \geq 1$, and $p_0 = 1$. Let $q_n = p_r(n)/n!$. By multiplying the recurrence by $x^{n-1}/(n-1)!$, and summing up over all $n \geq 1$, we obtain

$$\frac{d}{dx} q(x) = (r-1) \frac{d}{dx}(xq(x)) + \frac{q(x)}{1-(r-l)x},$$

where $q(x)$ is the generating function of q_n. Besides $q(0)=1$, hence the theorem holds. $\square$

Corollary 2. *For all $n \geq 0$, $|E_n^r(T_{1;1,2,\ldots,r})| = \prod\limits_{j=0}^{n}(j(r-1)+1)$.*

Proof. Immediately by the proof of Theorem 5, for $n \geq 2$

$$p_r(n) = |E_n^r(T_{1;1,2,\ldots,r})| = ((r-1)n+1)p_r(n-1).$$

Besides, $p_r(1)=r$, and $p_r(0)=1$, hence the corollary holds. $\square$

Example 7. (*see* [S, Eq. 47]) By Theorem 5, $|E_n^2((1^{(1)},2^{(1)}),(1^{(1)},2^{(2)}))| = (n+1)!$.

Now we represent the second special subset. Consider a subset $T \subset E_k^r$; we say that T is *good* if it is the union of disjoint homogeneous subsets; that is, $T = \cup_{j=1}^{p}(T_j)_{(u_j)}$. As an example, $T = \{123_{(1)}, 132_{(1)}, 213_{(2)}\}$ is a good set.

Theorem 6. *Let $T = \cup_{j=1}^{p}(T_j)_{(u_j)}$ be a good set. For $n \geq 0$*

$$|E_n^r(T)| = \sum_{j_1=0}^{n}\sum_{j_2=0}^{n-j_1}\cdots\sum_{j_p=0}^{n-j_1-\ldots-j_{p-1}}(r-p)^{n-j_1-\ldots-j_p}\frac{\binom{n}{j_1,j_2,\ldots,j_p}^2}{(n-j_1-\ldots-j_p)!}\prod_{i=1}^{p}|S_{j_i}(T_i)|.$$

Proof. The theorem holds for $p=1$ by Theorem 3. Now let $p>1$, so by definitions $|E_n^r(T)| = \sum_{j_1=0}^{n}|E_{n-j}^{1,\ldots,u_1-1,u_1+1,\ldots,r\}}(T\backslash(T_1)_{(u_1)})||S_j(T_1)|\binom{n}{j_1}^2$, therefore, $|E_n^r(T)| = \sum\limits_{j_1=0}^{n}|E_{n-j}^{r-1}(T\backslash(T_1)_{(u_1)})||S_j(T_1)|\binom{n}{j_1}^2$. Hence by induction the theorem holds. $\square$

Let $T_{d,l;a_1,\ldots,a_l}$ be a subset of E_2^k defined by

$$T_{d,l;a_1,\ldots,a_l} = \cup_{i=1}^{d}\{(1,2)_{(a_i)}\}\bigcup\cup_{i=d+1}^{l}\{(2,1)_{(a_i)}\},$$

hence by Theorem 6 we obtain the following corollary:

Corollary 3. *Let $1 \leq a_1,\ldots,a_l \leq k$ be l different numbers. For $n \geq 0$,*

$$|E_n^r(T_{d,l;a_1,\ldots,a_l})| = \sum_{i_1+\ldots+i_l\leq n}\frac{\binom{n}{i_1,\ldots,i_l}^2}{(n-i_1-\ldots-i_l)!}(r-l)^{n-i_1-\ldots-i_l}.$$

Now we built a bijection, which gives for the set $E_n^r(T_{d,a;a_1,\ldots,a_l})$ a combinatorial geometric explanation. Consider l lines $L_1,\ldots,L_l$ such that L_i contains all the points of the form $j^{(i)}$ for all $j=1,2,\ldots,n$. We say L_i is *good* if the points $1^{(i)}$ to $n^{(i)}$ are decreasing, and the line L_i is *bad* if the points $1^{(i)},\ldots,n^{(i)}$ are increasing, otherwise we say the line L_i is *free*.

Now we consider the following collection which represents the set $T_{d,l;a_1,\ldots,d_l}$. Let $L_{a_1},\ldots,L_{a_d}$ be good lines, $L_{a_{d+1}},\ldots,L_{a_l}$ be bad lines, and L_i be a free line for all $1 \leq i \leq k$ such that $i \notin \{a_1,\ldots,a_l\}$. For example, the representation of $T_{1,2;3,2}$ where $k=4$, is given by the following diagram.

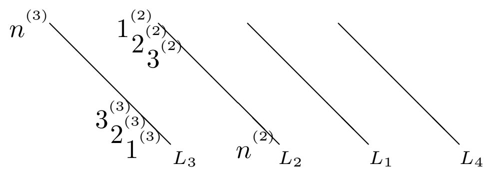

Figure 1: Representation of $T_{1,2;3,2}$

Here the lines L_1 and L_4 are free lines.

Now let us define a *path* between the points on the lines of the representation of $T_{d,l;a_1,\ldots,a_l}$. A path is a collection of steps, starting anywhere, such that every step is one of the following steps:

(i) a decreasing step from a point to another point on a bad, or a good line,
(ii) a free step on the free line, or between the lines (from a point to another point).

Hence by definitions we immediately have the following proposition.

Proposition 2. *Every path of n steps is a $T_{d,l;a_1,\ldots,a_l}$-avoiding signed permutation in E_n^r.*

Using the above proposition we find the cardinality of the set $E_n^r(T_{d,l;a_1,\ldots,a_l})$ by the following theorem.

Theorem 7. *Let $a_1,\ldots,a_l$ be l different numbers such that $1 \le a_i \le r$ for all $i = 1,2,\ldots,l$. For $n \ge 0$,*

$$|E_n^r(T_{d,l;a_1,\ldots,a_l})| = \sum_{i_1+\ldots+i_l \le n} \frac{\binom{n}{i_1,\ldots,i_l}^2}{(n-i_1-\ldots-i_l)!}(r-l)^{n-i_1-\ldots-i_l}.$$

Proof. To choose a path of n steps with l points in bad or good lines we have to:

(i) choose $i_1,\ldots i_l$ places in the path. There are $\binom{n}{i_1,\ldots,i_l}$ possibilities.
(ii) choose $i_1,\ldots,i_l$ points from bad or good lines. There are $\binom{n}{i_1,\ldots,i_l}$ possibilities.
(iii) choose $n-d$ points on free lines, where $d = i_1+\ldots+i_l$. There are $(n-d)!(k-l)^{n-d}$ possibilities.

Hence, by Proposition 2 the theorem holds □

By Theorem 7 we obtain a generalization of certain results in [S], particularly we get the following corollary.

Corollary 4. *Let $0 \le d \le r$; for $n \ge 0$,*

$$|E_n^r(T_{d,r;1,2,3,\ldots,r})| = \sum_{i_1=0}^{n}\sum_{i_2=0}^{n-i_1}\cdots\sum_{i_{r-1}=0}^{n-i_1-\ldots,i_{r-2}} \binom{n}{i_1,\ldots,i_{r-1},n-i_1-\ldots-i_{r-1}}^2.$$

Example 8. (*see* [S, Eq. 49]) By Corollary 4 we obtain for $\beta,\gamma \in S_2$

$$|E_n^2(\beta_{(1)},\gamma_{(2)})| = \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

5. Single restriction by a 2-letter signed pattern

The length 2 signed permutations give rise to some enumeratively interesting classes of signed permutations, which we examine in this section. In the symmetric group S_n, patterns of length 2 are uninterestingly restrictive, and length 3 is the first interesting case. Also in E_n^r, restriction by patterns of length 1 is trivial, and given by the following formula $|E_n^r(1^a)| = n!\cdot(r-1)^n$, where $1 \le a \le r$. Let us denote $d_r(n) = \sum_{j=0}^{n} j!(r-1)^j\binom{n}{j}^2$, and let $d_r(x)$ be the generating function of the sequence $d_r(n)/n!$. Hence it easy to see that $d_r(x) = \frac{e^{\frac{x}{1-(r-1)x}}}{1-(r-1)x}$.

Now we prove the first case of Main Theorem, that is, that there exists exactly one r-Wilf class of a single restriction by a 2-letter signed pattern, for all $r \ge 1$.

Theorem 8. *Let $r \ge 1$, and $1 \le a,b,c,d \le r$. For $n \ge 0$*

$$|E_n^r((1^{(a)},2^{(b)}))| = |E_n^r((2^{(c)},1^{(d)}))| = d_r(n).$$

Proof. By section 2 (symmetric operations) we have to prove the following two cases:

1. Let $1 \le a \le r$; for $n \ge 0$, $|E_n^r((1^{(a)},2^{(a)}))| = |E_n^r((2^{(a)},1^{(a)}))| = d_r(n)$;
2. Let $b \le a$; for $n \ge 0$, $|E_n^r((1^{(a)},2^{(b)}))| = |E_n^r((1^{(a)},2^{(a)}))|$.

The first, and the second cases are obtained immediately by Theorem 3, and Theorem 5, respectively. ☐

6. Double restrictions by 2-letter signed patterns

In this section, we find the number of r-Wilf clases, $r \ge 1$, of double restrictions by 2-letter signed patterns. In E_2^r there are $k^2(k^2-1)$ possibilities to choose two elements of the following form: $(1^{(a)},2^{(b)}),(1^{(c)},2^{(d)})$, and there are k^4 possibilities to choose two lements of the following form: $(1^{(a)},2^{(b)}),(2^{(c)},1^{(d)})$, where $1 \le a,b,c,d \le r$. On the other hand, by symmetric operations (section 2), the question of determining the $E_n^r([\tau]_a,[\tau']_{a'})$ for $k^2(2k^2-1)$ choices for 2-letter signed patterns $[\tau]_a$, $[\tau']_{a'}$ reduces to determining the $E_n^r([\tau]_a,[\tau']_{a'})$ where $[\tau]_a$, $[\tau']_{a'}$ are from Table 1.

Theorem 9. *For $n \ge 0$, $|E_n^r(T)| = n!(n+r-1)(r-1)^{n-1}$ where*

(i) $T = \{(1^{(1)},2^{(1)}),(2^{(1)},1^{(1)})\}$ for $r \ge 1$;

Case	$[\tau]_a$	$[\tau']_{a'}$	$\lvert E_n^5([\tau]_a,[\tau']_{a'})\rvert$ for $n=0,1,2,3,4,5$	Reference
1	$(1^{(1)},2^{(1)})$	$(2^{(1)},1^{(1)})$	$1,5,48,672,12288,276480$	Theorem 9
2	$(1^{(1)},2^{(1)})$	$(1^{(1)},2^{(2)})$	$1,5,48,672,12288,276480$	Theorem 9
3	$(1^{(1)},2^{(2)})$	$(2^{(1)},1^{(2)})$	$1,5,48,672,12288,276480$	Theorem 9
4	$(1^{(1)},2^{(2)})$	$(2^{(2)},1^{(1)})$	$1,5,48,672,12288,276480$	Theorem 9
5	$(1^{(1)},2^{(2)})$	$(1^{(1)},2^{(3)})$	$1,5,48,672,12288,276480$	Theorem 9
6	$(1^{(1)},2^{(1)})$	$(1^{(2)},2^{(2)})$	$1,5,48,668,12046,265062$	Theorem 10
7	$(1^{(1)},2^{(1)})$	$(2^{(2)},1^{(2)})$	$1,5,48,668,12046,265062$	Theorem 10
8	$(1^{(1)},2^{(1)})$	$(1^{(2)},2^{(3)})$	$1,5,48,668,12046,265062$	Theorem 10
9	$(1^{(1)},2^{(1)})$	$(2^{(2)},1^{(3)})$	$1,5,48,668,12046,265062$	Theorem 10
10	$(1^{(1)},2^{(2)})$	$(1^{(3)},2^{(4)})$	$1,5,48,668,12046,265062$	Theorem 10
11	$(1^{(1)},2^{(2)})$	$(2^{(3)},1^{(4)})$	$1,5,48,668,12046,265062$	Theorem 10
12	$(1^{(1)},2^{(2)})$	$(2^{(1)},1^{(3)})$	$1,5,48,670,12168,270856$	Theorem 11
13	$(1^{(1)},2^{(2)})$	$(2^{(2)},1^{(3)})$	$1,5,48,670,12168,270856$	Theorem 11
14	$(1^{(1)},2^{(2)})$	$(2^{(3)},1^{(1)})$	$1,5,48,670,12168,270856$	Theorem 11
15	$(1^{(1)},2^{(1)})$	$(2^{(1)},1^{(2)})$	$1,5,48,671,12288,273665$	Theorem 12
16	$(1^{(1)},2^{(2)})$	$(1^{(2)},2^{(3)})$	$1,5,48,669,12106,267867$	
17	$(1^{(1)},2^{(2)})$	$(1^{(2)},2^{(1)})$	$1,5,48,670,12166,270672$	

Table 1. Pairs of 2-letter signed patterns

(ii) $T=\{(1^{(1)},2^{(1)}),(1^{(1)},2^{(2)})\}$ *for* $r\geq 2$*;*
(iii) $T=\{(1^{(1)},2^{(2)}),(2^{(1)},1^{(2)})\}$ *for* $r\geq 2$*;*
(iv) $T=\{(1^{(1)},2^{(2)}),(2^{(2)},1^{(1)})\}$ *for* $r\geq 2$*;*
(v) $T=\{(1^{(1)},2^{(2)}),(1^{(1)},2^{(3)})\}$ *for* $r\geq 3$.

Proof. By Theorem 3 it is easy to obtain (i), and Theorem 5 immediately yields (ii), and (v). Now let us prove (iii) and (iv).

Case (iii)**:** Let $p_n=|E_n^r(T)|$, $\Phi\in E_n^r(T)$, and let us consider the possible values of Φ_1:

1. Let $\Phi_1=i^{(c)}$, $c\neq 1$; $\Phi\in E_n^r(T)$ if and only if $(\Phi_2,\dots,\Phi_n)\in E^r_{\{1,\dots,i-1,i+1,\dots,n\}}(T)$. Hence in this case there are $(r-1)np_{n-1}$ signed permutations.
2. Let $\Phi_1=i^{(1)}$; since Φ is T-avoiding, the symbols $1,\dots,i-1,i+1,\dots,n$ are not signed by 2, and can be replaced anywhere at positions $2,\dots,n$. Hence, in this case there are $(n-1)!(r-1)^{n-1}$ signed permutations.

Therefore by the above three cases p_n staisfies the following relation:

$$p_n=n(r-1)p_{n-1}+n!(r-1)^{n-1}.$$

Besides $p_0=1$, and $p_1=r$, hence (iv) holds.

Case (iv)**:** Let $p_n=|E_n^r(T)|$, $\Phi\in E_n^r(T)$ such that $\Phi_j=n^{(c)}$, and let us consider the possible values of j,c:

1. Let $c \neq 2$; $\Phi \in E_n^r(T)$ if and only if $(\Phi_1, \ldots, \Phi_{j-1}, \Phi_{j+1}, \ldots, \Phi_n) \in E_{n-1}^r(T)$. Hence in this case there are $(r-1)np_{n-1}$ signed permutations.
2. Let $c = 2$; $\Phi \in E_n^r(T)$ if and only if $(\Phi_1, \ldots, \Phi_{j-1}, \Phi_{j+1}, \ldots, \Phi_n)$ is a signed permutation with symbols $1, 2, \ldots, n-1$ and signs $2, \ldots, r$. Hence, in this case there are $(n-1)!(r-1)^{n-1}$ signed permutations.

Therefore by the above three cases p_n staisfies the following relation:

$$p_n = n(r-1)p_{n-1} + n!(r-1)^{n-1}.$$

Besides $p_0 = 1$, and $p_1 = r$, hence (v) holds. $\square$

Example 9. (*see* [S, Eq. 46, 47]) As an example we get

$$\begin{aligned}&|E_n^2((1^{(1)}, 2^{(1)}), (2^{(1)}, 1^{(1)}))| = |E_n^2((1^{(1)}, 2^{(1)}), (1^{(1)}, 2^{(2)}))| = \\ &|E_n^2((1^{(1)}, 2^{(2)}), (2^{(1)}, 1^{(2)}))| = |E_n^2((1^{(1)}, 2^{(2)}), (2^{(2)}, 1^{(1)}))| = (n+1)!\end{aligned}$$

for $n \geq 0$, which was proved in [S].

Theorem 10. *Let $2 \leq a \leq b$, and $r \geq b$; for all $n \geq 1$*

$$|E_n^r(T)| = \sum_{i+j \leq n} \binom{n}{i, j, n-i-j}^2 (n-i-j)!(r-2)^{n-i-j},$$

where

(i) $T = \{(1^{(1)}, 2^{(1)}), (1^{(a)}, 2^{(b)})\}$*;*
(ii) $T = \{(1^{(1)}, 2^{(1)}), (2^{(a)}, 1^{(b)})\}$*;*
(iii) $T = \{(1^{(1)}, 2^{(2)}), (1^{(3)}, 2^{(4)})\}$*;*
(iv) $T = \{(1^{(1)}, 2^{(2)}), (2^{(3)}, 1^{(4)})\}$*.*

Proof. **Cases** (i)**,** (ii)**:** Immediately by the proof of Theorem 6, and by part (i) of Main Theorem, we claim these cases.

Cases (iii)**,** (iv)**:** Let $T_1 = \{(1^{(1)}, 2^{(2)}), (1^{(3)}, 2^{(4)})\}$, $T_2 = \{(1^{(1)}, 2^{(2)}), (1^{(3)}, 2^{(4)})\}$, and let $\Phi \in E_n^r(T_1)$. Also let us define I_Φ to be the set of all j such that Φ_j is signed by either 3 or 4. Now we define a function $f : E_n^r(T_1) \to E_n^r(T_2)$ by reversing all the Φ_j where $j \in I_\Phi$. Hence by definitions, f is a bijection, which means that $|E_n^r((1^{(1)}, 2^{(2)}), (2^{(3)}, 1^{(4)}))| = |E_n^r((1^{(1)}, 2^{(2)}), (2^{(4)}, 1^{(3)}))|$.

On the other hand, immediately by Main Theorem part (i) we get

$$|E_n^r((1^{(1)}, 2^{(2)}), (2^{(3)}, 1^{(4)}))| = |E_n^r((1^{(1)}, 2^{(1)}), (2^{(4)}, 1^{(4)}))|,$$

which means by case (i) that the theorem holds. $\square$

Example 10. (*see* [S, Eq. 47]) As an example, by Theorem 10 for $n \geq 0$

$$|E_n^2((1^{(1)}, 2^{(1)}), (2^{(2)}, 1^{(2)}))| = \binom{2n}{n}.$$

Theorem 11. *For* $r \geq 3$, $\sum_{n\geq 0} \frac{E_n^r(T)}{n!} x^n = \frac{\int d_{r-1}^2(x)dx}{1-(r-1)x}$ *where*

(i) $T = \{(1^{(1)}, 2^{(2)}), (2^{(1)}, 1^{(3)})\}$;
(ii) $T = \{(1^{(1)}, 2^{(2)}), (2^{(2)}, 1^{(3)})\}$;
(iii) $T = \{(1^{(1)}, 2^{(2)}), (2^{(3)}, 1^{(1)})\}$.

Proof. **Case** (i)**:** Let $T = \{(1^{(1)}, 2^{(2)}), (2^{(1)}, 1^{(3)}))\}$, $p_n = E_n^r(T)$, $\Phi \in E_n^r(T)$, and let us consider the possible values of Φ_1:

1. Let $\Phi_1 = i^{(c)}$, $c \neq 1$; so $\Phi \in E_n^r(T)$ if and only if $(\Phi_2, \dots, \Phi_n) \in E^r_{\{1,\dots,i-1,i+1,\dots,n\}}(T)$. Hence in this case there are $(r-1)np_{n-1}$ signed permutations.
2. Let $\Phi_1 = i^{(1)}$; since Φ is T-avoiding, the symbols $i+1, \dots, n$ are not signed by 2, and the symbols $1, \dots, i-1$ are not signed by 3. Hence there are $\binom{n-1}{i-1}|E_{n-i}^{r-1}((2^{(1)}, 1^{(3)}))||E_{i-1}^{r-1}((1^{(1)}, 2^{(2)}))|$ signed permutations, which means by part (i) of Main Theorem that there are $\binom{n-1}{i-1}d_{r-1}(n-i)d_{r-1}(i-1)$ signed permutations.

Therefore p_n staisfies the following relation:

$$p_n = n(r-1)p_{n-1} + \sum_{i=1}^{n} \binom{n-1}{i-1} d_{r-1}(n-i)d_{r-1}(i-1).$$

Besides $p_0 = 1$, and $p_1 = r$, hence case (i) holds.

Case (ii)**:** Let $T = \{(1^{(1)}, 2^{(2)}), (2^{(2)}, 1^{(3)}))\}$, $p_n = E_n^r(T)$, and $\Phi \in E_n^r(T)$ such that $\Phi_j = n^{(c)}$. Let us consider the possible values of j, c:

1. Let $c \neq 2$; $\Phi \in E_n^r(T)$ if and only if $(\Phi_1, \dots, \Phi_{j-1}, \Phi_{j+1}, \dots, \Phi_n) \in E_{n-1}^r(T)$. Hence in this case there are $(r-1)np_{n-1}$ signed permutations.
2. Let $c = 2$; since Φ is T-avoiding, all the symbols in $(\Phi_1, \dots, \Phi_{j-1})$ are not signed by 1, and the symbols in $(\Phi_{j+1}, \dots, \Phi_n)$ are not signed by 3. Hence there are $\binom{n-1}{j-1}|E_{n-j}^{r-1}((2^{(2)}, 1^{(3)}))||E_{j-1}^{r-1}((1^{(1)}, 2^{(2)}))|$ signed permutations, which means by part (i) of Main Theorem that there are $\binom{n-1}{j-1}d_{r-1}(n-j)d_{r-1}(j-1)$ signed permutations.

Therefore p_n staisfies the following relation:

$$p_n = n(r-1)p_{n-1} + \sum_{j=1}^{n} \binom{n-1}{j-1} d_{r-1}(n-j)d_{r-1}(j-1).$$

Besides $p_0 = 1$, and $p_1 = r$, hence case (ii) holds.

Case (iii)**:** Similarly to the case (ii) for $\Phi \in E_n^r(T)$ such that $\Phi_j = 1^{(c)}$, we consider the possible values of j, c, and get the same result. □

Theorem 12. *For $r \geq 2$, $\sum_{n\geq 0} \frac{E_n^r((1^{(1)},2^{(1)}),(2^{(1)},1^{(2)}))}{n!}x^n = \frac{\int \frac{d_{r-1}(x)}{1-(r-1)x}dx}{1-(r-1)x}$.*

Proof. Let $T = \{(1^{(1)}, 2^{(1)}), (2^{(1)}, 1^{(2)}))\}$, $p_n = E_n^r(T)$, $\Phi \in E_n^r(T)$, and let us consider the possible values of Φ_1:

1. If $\Phi_1 = i^{(c)}$ where $c \neq 1$, then $\Phi \in E_n^r(T)$ if and only if $(\Phi_2, \ldots, \Phi_n) \in E^r_{\{1,\ldots,i-1,i+1,\ldots,n\}}(T)$. Hence in this case there are $(r-1)np_{n-1}$ signed permutations.
2. If $\Phi_1 = i^{(1)}$ then, since Φ avoids T, the symbols $i+1, \ldots, n$ are not signed by 1, and the symbols $1, \ldots, i-1$ are not signed by 2. Hence there are $|E^{r-1}_{n-i}||E^{r-1}_{i-1}((1^{(1)}, 2^{(1)}))|$ signed permutations, which means by part (i) of Main Theorem that there are $\binom{n-1}{i-1}(n-i)!(r-1)^{n-i}d_{r-1}(i-1)$ signed permutations.

Therefore p_n staisfies the following relation:

$$p_n = n(r-1)p_{n-1} + \sum_{i=1}^{n} \binom{n-1}{i-1}(n-i)!(r-1)^{n-i}d_{r-1}(i-1).$$

Besides $p_0 = 1$, and $p_1 = r$, hence the theorem holds. □

Example 11. (*see* [S, Eq. 48]) Let us denote $a_n = |E_n^2((1^{(1)}, 2^{(1)}), (2^{(2)}, 1^{(1)}))|$; by symmetric operations and by Theorem 12, $a_n = na_{n-1} + (n-1)! \sum_{j=0}^{n-1} \frac{1}{j!}$ for $n \geq 1$, hence $n! < p_n < (n+1)!$ for $n \geq 3$.

Corollary 5. *Let $wc(r)$ be the number of r-Wilf classes of a double restriction by 2-letter signed patterns. Then for $r \geq 1$*

$$wc(r) = \begin{cases} 1\ , & \text{if } r = 1 \\ 4\ , & \text{if } r = 2 \\ 6\ , & \text{if } r \geq 3 \end{cases}.$$

Proof. By Theorem 9, Theorem 10, Theorem 11, and Theorem 12 we get $wc(1) = 1$, $wc(2) = 4$. The rest follows from definitions, and by the first elements of $E_n^3(T)$ where T set of two signed patterns from Table 1. □

7. Combinatorial identity

First of all let us define for $a_1 \leq a_2 \leq \ldots \leq a_{2l}$

$$U^{b_1,\ldots,b_l}_{a_1,\ldots,a_{2l}} = \{(1^{(a_{2i-1})}, 2^{(a_{2i})})|b_i = 0\} \cup \{(2^{(a_{2i-1})}, 1^{(a_{2i})})|b_i = 1\},$$

where either $b_i = 0$, or $b_i = 1$ for $i = 1, 2, \ldots, l$. So by part (i) of the Main Theorem we obtain the following corollary.

Corollary 6. *Let* $1 \le a_1 \le \ldots \le a_{2l} \le r$*, and* $b_i \in \{0,1\}$ *for all* $i = 1, 2, \ldots, l$*. Then* $|E_n^r(U_{a_1,\ldots,a_{2l}}^{b_1,\ldots,b_l})| = |E_n^r(U_{1,1,2,2,\ldots,l,l}^{0,\ldots,0})|$.

By Theorem 6 $|E_n^r(U_{1,1,2,2,\ldots,l,l}^{0,\ldots,0})|$ is equal to

$$\sum_{i_1+\ldots+i_l \le n} \binom{n}{i_1, \ldots, i_l, n-i_1-\ldots-i_l}^2 (n-i_1-\ldots-i_l)!(r-l)^{n-i_1-\ldots-i_l}.$$

On the other hand, by part (i) of Main Theorem $|E_n^r(U_{1,2,3,4,\ldots,2l}^{0,\ldots,0})|$ is equal to

$$\sum_{i_1+\ldots+i_l \le n} \binom{n}{i_1, \ldots, i_l, n-i_1-\ldots-i_l}^2 (n-i_1-\ldots-i_l)!(r-2l)^{n-i_1-\ldots-i_l} \prod_{j=1}^{l} d_2(i_j).$$

where $d_2(m) = \sum j!\binom{m}{j}^2$. Hence by the above Corollary we obtain the following theorem.

Theorem 13. *Let* $r \ge 2l$*. For* $n \ge 0$

$$\sum_{i_1+\ldots+i_l \le n} \frac{\binom{n}{i_1,\ldots,i_l}^2}{(n-i_1-\ldots-i_l)!}(r-l)^{n-i_1-\ldots-i_l} =$$
$$= \sum_{i_1+\ldots+i_l \le n} \frac{\binom{n}{i_1,\ldots,i_l}^2}{(n-i_1-\ldots-i_l)!}(r-2l)^{n-i_1-\ldots-i_l} \prod_{j=1}^{l} d_2(i_j).$$

Example 12. By Theorem 13 for $l = 1$, and $r = 3$, we obtain

$$\sum_{i=0}^{n} \frac{2^{n-i}}{i!^2(n-i)!} = \sum_{i=0}^{n}\sum_{j=0}^{i} \frac{1}{j!(i-j)!^2(n-i)!}.$$

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